(1) $$\sin 2\beta = \sin AOC = \sin ACD \cdot \frac{AC}{R} = \frac{AC \cdot AD}{2R^2} = 2\frac{AC}{2R} \cdot \frac{AD}{2R} = 2 \sin \beta \cos \beta.$$

(2) $$\cos 2\beta = \cos AOC = \frac{R^2 + R^2 - \overline{AC}^2}{2R \cdot R} = \frac{4R^2 - \overline{AC}^2 - \overline{AC}^2}{4R^2} = \frac{\overline{AD}^2 - \overline{AC}^2}{4R^2}$$

$$= \left(\frac{AD}{2R}\right)^2 - \left(\frac{AC}{2R}\right)^2 = \cos^2 \beta - \sin^2 \beta.$$

(3) $$\sin 3\beta \sin AEC = \sin AEO = \sin \beta \cdot \frac{R}{OE}.$$

Now

$$\frac{OE}{R} = \frac{AE}{AD} = \frac{AE}{2R \cos \beta}; \quad i.\ e.,\ AE = 2\overline{OE} \cos \beta.$$

Again

$$OE^2 = \overline{AE}^2 + R^2 - 2\overline{AE} \cdot R \cos \beta = 4\overline{OE}^2 \cos^2 \beta + R^2 - 4\overline{OE} \cdot R \cdot \cos^2 \beta;$$

$$\therefore\ OE = \frac{R}{4 \cos^2 \beta - 1};$$

$$\therefore\ \sin 3\beta = \sin \beta (4 \cos^2 \beta - 1) = 3 \sin \beta - 4 \sin^3 \beta.$$

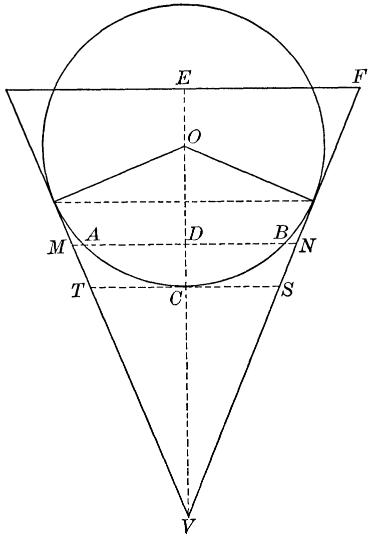
(4) $$\cos 3\beta = \frac{R^2 - \overline{AE}^2 - \overline{OE}^2}{2\overline{OE} \cdot AE} = \frac{(4 \cos^2 \beta - 1)^2 - 1 - 4 \cos^2 \beta}{4 \cos \beta} = 4 \cos^3 \beta - 3 \cos \beta.$$

462. Proposed by DANIEL KRETH, Wellman, Iowa.

A conical glass, the diameter of the base of which is 5 inches and altitude 6 inches, is one-fifth full of water. If a sphere 4 inches in diameter is dropped into it, how much of the vertical axis of the glass is immersed?

SOLUTION BY J. A. CAPRON, Notre Dame University.

Let MN be the level of the water after the sphere is dropped; $R = 5/2$ inches = radius of base of given cone; $r = 2$ inches = radius of sphere; r_1 = radius of base of cone VMN; x = height of spherical segment ACB; α = semivertical angle of cone; and $CV = a$.



Then volume of cone $VMN = V_c = \frac{1}{3}\pi r_1^2(a+x)$; volume of segment $ACB = V_s = \pi x^2(r - x/3)$; and volume of given cone $= V = \frac{1}{3}\pi R^2 h$, where $h = 6$ inches.

Hence $V_c - V_s = \frac{1}{5}V$, since $V_c - V_s =$ volume of water. Substituting, we get

$$r_1^2(a + x) - x^2(3r - x) = \frac{R^2h}{5}.$$

Since

$$VC = VO - CO \qquad \text{and} \qquad VO = \frac{r}{\sin\alpha},$$

we get

$$a = r\left(\frac{1}{\sin\alpha} - 1\right),$$

but since

$$\tan\alpha = \frac{5}{12}, \qquad \sin\alpha = \frac{5}{13},$$

therefore

$$a = 2\left(\frac{13}{5} - 1\right) = \frac{16}{5}.$$

From triangles VEF and VDN, we get

$$r_1 = \frac{(a + x)}{h}R, \qquad \text{or} \qquad r_1 = \frac{\left(\frac{16}{5} + x\right)\frac{5}{2}}{6} = \frac{4}{3} + \frac{5x}{12}.$$

Substituting these values in the above equation and reducing, we get

$$845x^3 - 3120x^2 + 3840x - 1304 = 0.$$

To solve this equation, let

$$x = y + \frac{108}{169}, \qquad \text{or} \qquad x = y + \frac{16}{13}.$$

Substituting and reducing, we get

$$169y^3 = -\frac{16^3}{13} + 260.8, \qquad \text{or} \qquad 13^3y^3 = -705.6.$$

Hence,

$$y = -\frac{\sqrt[3]{705.6}}{13} = -\frac{2}{13}\sqrt[3]{88.2}.$$

Then

$$x = y + \frac{16}{13} = \frac{16 - 2\sqrt[3]{88.2}}{13} = .54595.$$

Hence, the required height $= a + x = 3.2 + .54595 = 3.74595$ inches.

Excellent solutions were received from NATHAN ALTSHILLER, C. N. SCHMALL, HERBERT N. CARLETON, J. W. CLAWSON, HORACE OLSON, and PAUL CAPRON.

CALCULUS.

375. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the differential equation

$$x^2(a - bx)\frac{d^2y}{ax^2} - 2x(2a - bx)\frac{dy}{dx} + 2(3a - bx)y = 6a^2.$$

I. SOLUTION BY H. T. BIGELOW, La Fayette, Indiana.

Let

$$y = \sum_{n=0}^{\infty} c_n x^n, \qquad \frac{dy}{dx} = \sum_{n=0}^{\infty} nc_n x^n, \qquad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n - 1)c_n x^{n-2}.$$

Substituting and collecting terms we have,

$$\sum_{n=0}^{\infty} a(n^2 - 5n + 6)c_n x^n \equiv \sum_{n=0}^{\infty} b(n^2 - 3n + 2)c_n x^{n+1} + 6a^2.$$